Non-orthogonal elastoplastic model for sand with dilatancy

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ABSTRACT: A non-orthogonal elastoplastic constitutive model for the sand with dilatancy is presented in the characteristic stress space. Dilatancy of sand is represented by the direction of plastic flow, which can be directly determined by applying the non-orthogonal plastic flow rule to an improved elliptic yield function. A new hardening parameter is developed to describe the contractive and dilative volume change during the shear process, which is coordinated with the non-orthogonal plastic flow direction. The combination of the non-orthogonal plastic flow rule and the proposed hardening parameter renders the proposed model with the ability to reasonably describe the stress-strain relationship of sand with dilatancy. The model performance is evaluated by comparing with the experimental data of sand under triaxial conditions.

Keywords: Sand; Dilatancy; Constitutive model; Non-orthogonal plastic flow rule; Hardening parameter; Characteristic stress

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1 Introduction

Dilatancy refers to the volume change observed in granular materials during shear deformations [1-4]. It is an essential property of sand. Loose sand tends to show negative dilatancy of volumetric contraction, while dense sand undergoes positive dilatancy [4, 5]. One of the fundamental issues in modelling the mechanical behaviour of sand is to describe the dilatancy by the ratio of the plastic volumetric strain increment to the plastic shear strain increment [6, 7]. The ratio can be expressed by the direction of the plastic flow [8, 9]. Since the axis of the plastic strain increment is assumed to be coaxial with that of stress, the plastic flow direction can be determined by the plastic flow rule in the stress space [10-14]. The direction of the plastic strain increment is usually determined based on the orthogonal gradient of the plastic potential function. Obviously, this makes the plastic flow direction and its evolution rule inseparable from the plastic potential surface. However, an appropriate plastic potential function is difficult to be directly constructed based on experimental data, and some additional parameters may be required [15-17]. In fact, the purpose of constructing the plastic potential function is to produce the proper plastic flow direction. As an alternative, the plastic flow direction can also be directly achieved by applying the non-orthogonal plastic flow rule to the yield function [18-20]. The non-orthogonal plastic flow rule has been used to model the dilatancy behaviours of soil under triaxial compression conditions [20-23]. However, for three-dimensional (3D) stress conditions, the non-orthogonal plastic flow rule has not been well applied to simulate positive dilatancy [19]. How to use the non-orthogonal plastic flow rule to describe the negative/positive dilatancy of sand under 3D stress conditions and then establish a 3D non-orthogonal model for sand is worthy of further study.

Although the non-orthogonal plastic flow direction can be used to represent the stress-dilatancy relationship for sand, it does not mean that the shear responses of the sand with dilatancy have been totally captured. A proper hardening parameter is still needed in describing the contractive-dilative behaviour of sand. If hardening rule stems only from volume change, just as the models for normally consolidated clay [21, 24-26], once the critical state of zero incremental dilation is reached, the ultimate plastic deformation occurs and hardening stops [17]. Obviously, the plastic volumetric strain is not suitable for direct use as a hardening parameter to predict the dilative behaviour of sand. Therefore, the hardening parameter should be reconstructed. Several hardening parameters and hardening modulus have been proposed and used for modelling dilatancy behaviours of sand that is related to

density and stress state [27-30], anisotropy [31-33] and even fabric evolution [34-36]. However, they are all formulated based on orthogonal plastic flow conditions. As a key point to reflect the dilatancy of sand, an appropriate hardening parameter should be coordinated with the plastic flow direction. That is, the hardening parameters derived for the orthogonal plastic flow rule cannot be directly introduced into a non-orthogonal model. Therefore, a hardening parameter should be reconstructed to match up with the non-orthogonal plastic flow direction.

In this paper, the non-orthogonal plastic flow rule is applied to capture the dilatancy behaviour of sand. The plastic flow direction under 3D stress conditions is directly determined by applying the non-orthogonal plastic flow rule to the elliptic yield function in the characteristic stress space. Furthermore, a hardening parameter for sand with dilatancy is constructed to match up with the non-orthogonal plastic flow direction. The capability of the proposed 3D model in modelling the shear response of sand with the negative and positive dilatancy is verified by predicting test results of sand.

2 Framework of 3D non-orthogonal elastoplastic model in the c_{ij} space

Based on the elastoplastic theory, the total strain ε_{ij} and its increment $d\varepsilon_{ij}$ can be decomposed into the reversible elastic strain component and the irreversible plastic strain component:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p, \quad d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \quad (1)$$

For the plastic strain increment, the non-orthogonal plastic flow rule proposed by Lu *et al.* [19, 37] is adopted and can be expressed as:

$$d\varepsilon_{ij}^p = \Lambda \cdot m_{ij} = \Lambda \cdot \frac{\partial^\mu f}{\partial S_{kl}^\mu} \frac{\partial S_{kl}}{\partial \sigma_{ij}} \quad (2)$$

where Λ is the plastic multiplier, and m_{ij} ($i, j=1, 2, 3$) is the plastic flow direction. S_{kl} ($k, l=1, 2, 3$) is stress invariant that constructs the yield function and is associated with σ_{ij} . The non-orthogonal plastic flow rule is illustrated in Fig. 1: first determine the non-orthogonal gradient $\partial^\mu f / \partial S_{kl}^\mu$ of the yield function by the fractional derivative in the S_{kl} coordinate system, and then transform the non-orthogonal gradient in the S_{kl} coordinate system to the σ_{ij} coordinate system by the coordinate transformation $\partial S_{kl} / \partial \sigma_{ij}$. The non-orthogonal plastic flow rule in tensor form is the kernel for determining the general expression of plastic strain increment, which is essential for the model implementation [26, 38, 39]. The plastic flow direction m_{ij} is non-orthogonal to the yield surface, which

87 can also degenerate into the normal direction n_{ij} when $\mu=1$.

88
89 **Fig. 1** Schematic diagram of the non-orthogonal plastic flow rule.

90 In order to make the proposed non-orthogonal model be born with the potential to reflect the
91 3D properties of soil, the concept of the characteristic stress c_{ij} is introduced [19, 40]. More details on
92 the definition of the c_{ij} and the corresponding variables are provided in Appendix A. The general
93 form of the yield function in the c_{ij} space can thus be expressed as:

$$94 \quad f = f[c_{ij}, H(\varepsilon_{rs}^p)] = 0 \quad (3)$$

95 where H is the hardening parameter indicating the hardening degree of soil due to the occurrence of
96 plastic strain ε_{rs}^p ($r, s=1, 2, 3$). In view of the superior abilities of c_{ij} in reflecting 3D properties of
97 soil only through two characteristic stress invariants c_n and c_s [19], the yield function can be written
98 as:

$$99 \quad f = f[c_n, c_s, H(\varepsilon_{rs}^p)] = 0 \quad (4)$$

100 where c_n and c_s are the mean characteristic principal stress and the characteristic shear stress,
101 respectively.

102 By applying the non-orthogonal plastic flow rule in Eq. (2) to the yield function in Eq. (4),
103 the plastic flow direction m_{ij} in tensor form can be derived in the c_{ij} space:

$$104 \quad m_{ij} = \frac{\partial^\mu f}{\partial c_n^\mu} \frac{\partial c_n}{\partial c_{ij}} + \frac{\partial^\mu f}{\partial c_s^\mu} \frac{\partial c_s}{\partial c_{ij}} \quad (5)$$

105 The plastic multiplier Λ can be determined by the consistency condition, which is the total
106 differential of the yield function $f=0$ in the c_{ij} space:

$$df = \frac{\partial f}{\partial c_{ij}} dc_{ij} + \frac{\partial f}{\partial H} \frac{\partial H}{\partial \varepsilon_{rs}^p} d\varepsilon_{rs}^p = 0 \quad (6)$$

Where ε_{rs}^p is the plastic strain used for constructing the hardening parameter H . By substituting

$d\varepsilon_{rs}^p = \Lambda \cdot m_{rs}$ from Eq. (2) into Eq. (6), Λ can thus be expressed as:

$$\Lambda = - \frac{\frac{\partial f}{\partial c_{ij}} dc_{ij}}{\frac{\partial f}{\partial H} \frac{\partial H}{\partial \varepsilon_{rs}^p} m_{rs}} \quad (7)$$

By substituting the plastic flow direction m_{ij} in Eq. (5) and the plastic multiplier Λ in Eq. (7)

into Eq. (2), the plastic strain increment in the tensor form can be written as:

$$d\varepsilon_{ij}^p = \Lambda \cdot m_{ij} = - \frac{\frac{\partial f}{\partial c_{ij}} dc_{ij}}{\frac{\partial f}{\partial H} \frac{\partial H}{\partial \varepsilon_{rs}^p} m_{rs}} \cdot \left(\frac{\partial^\mu f}{\partial c_n^\mu} \frac{\partial c_n}{\partial c_{ij}} + \frac{\partial^\mu f}{\partial c_s^\mu} \frac{\partial c_s}{\partial c_{ij}} \right) \quad (8)$$

For the elastic strain increment in Eq. (1), we can make use of the elastic theory, e.g. Hooke's law:

$$d\varepsilon_{ij}^e = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} \quad (9)$$

where ν is the Poisson's ratio. E is the Young's modulus, which can be obtained from the e - $\ln p$ relationship of the swelling line under the isotropic compression conditions:

$$E = 3(1-2\nu) \frac{1+e_0}{\kappa} p \quad (10)$$

where κ is the swelling index, and e_0 is the initial void ratio at $p=p_0$.

Finally, by substituting Eqs. (9) and (8) into Eq. (1), the total strain increment is thus determined within the framework of the 3D non-orthogonal elastoplastic model.

3 Formulation of 3D non-orthogonal model

Within the framework of the non-orthogonal plastic theory, the yield function plays the key role in determining Λ and m_{ij} . In this paper, the improved elliptic yield function in the c_{ij} space is adopted.

$$f = c_s^2 + N^2 (c_n^2 - c_{nx} c_n) = 0 \quad (11)$$

where N is the ratio of the vertical axis to the horizontal axis of the elliptic yield curve as shown in

Fig. 2, which can be determined by the method in Section 3.4 based on the phase transition stress ratio F . The equivalent characteristic consolidation pressure c_{nr} is the intersection of f with the mean stress axis and defines the position of the yield surface, which can be expressed by the current stress state as follows:

$$c_{nr} = \frac{c_s^2 + N^2 c_n^2}{N^2 c_n} \quad (12)$$

Fig. 2 Yield curve in the characteristic stress space.

Note that our yield function can also reduce to the yield function of the modified Cam-clay model [25] when $\beta=1$ and $N=M$. As shown by Lu *et al.* [19], the axisymmetric ellipsoidal yield surface in the c_{ij} space corresponds to a smooth and convex triaxial symmetric yield surface in the σ_{ij} space. That is, in the σ_{ij} space, the asymmetric curve with respect to the p -axis in the meridian plane and the curved triangle in the deviatoric plane can both be directly described by the simple expression in the c_{ij} space (i.e., Eq. (11)). Therefore, the θ effect and stress-induced anisotropy of geomaterials can be reasonably captured by the yield function.

3.1 3D non-orthogonal plastic flow direction

In this paper, the non-orthogonal plastic flow direction is expressed based on the Caputo fractional derivative operator. The specific expressions and properties of the Caputo fractional derivative operator involved in this paper are provided in Appendix B.

By applying the non-orthogonal plastic flow rule in Eq. (2) to the yield function in Eq. (11), and further incorporating Eq. (12) and the recursive properties of the Gamma function $\Gamma(x+1)=x\cdot\Gamma(x)$, the non-orthogonal gradient $\mathbf{m}=(m_n, m_s)$ of the yield function can be expressed as:

$$\mathbf{m} = (m_n, m_s) = \left(\frac{\mu N^2 c_n^2 + (\mu - 2) c_s^2}{\Gamma(3 - \mu) c_n^\mu}, \frac{2 c_s^{2-\mu}}{\Gamma(3 - \mu)} \right) \quad (13)$$

Furthermore, by combining Eq. (13) with the coordinate transformation coefficient $\partial c_n / \partial c_{ij} = \delta_{ij} / 3$ and $\partial c_s / \partial c_{ij} = 3/2 \cdot (c_{ij} - c_n \delta_{ij}) / c_s$, the plastic flow direction tensor m_{ij} can thus be obtained:

$$m_{ij} = \frac{\mu N^2 c_n^2 + (\mu - 2) c_s^2}{\Gamma(3 - \mu) c_n^\mu} \frac{\delta_{ij}}{3} + \frac{2 c_s^{2-\mu}}{\Gamma(3 - \mu)} \frac{3(c_{ij} - c_n \delta_{ij})}{2 c_s} \quad (14)$$

3.1.1 Plastic flow direction in the c_{ij} space

The evolution of the plastic flow direction with the expansion of the yield surface will be illustrated, and its capability to model the dilatancy of sand will be explored as follows. To gain perspectives, let us consider the following material parameters for demonstration purpose: $M=0.95$, $M_f=1.65$ and $\beta=0.4$, which are typical for dense sand. Therefore, the phase transition stress ratio $F=0.366$ and the failure stress ratio $F_f=0.664$ are correspondingly obtained based on expressions listed in Tab. A-1 (see Appendix A), and $N=0.62$ and $\mu=0.52$ can be determined by the method in Section 3.4. Typical stress paths under true triaxial conditions under $p=196\text{kPa}$ at different stress Lode angles θ ($\theta=0^\circ, 15^\circ, 30^\circ, 45^\circ$ and 60°) are used for illustrating the variation rule of the plastic flow direction.

The simple expression of the yield function makes the yield surface in a simple ellipsoidal shape in the c_{ij} space, and the ellipsoidal yield surface expands only when the size changes without changing shape, as shown in Fig. 3. The elliptic yield curve in the meridian plane and the circular yield curve in the deviatoric plane are shown in Fig. 4(a) and (b), respectively. With the expansion of the ellipsoidal yield surface, the plastic flow directions for $\mu=0.52$ and $\mu=1.00$ correspondingly change, as shown in Fig. 3 and Fig. 4. The plastic flow direction non-orthogonal to the yield surface when $\mu=0.52$ is represented by the red arrow, and the plastic flow direction orthogonal to the yield surface when $\mu=1.00$ is represented by the black arrow. As the shear stress increases in the meridian plane, the non-orthogonal plastic flow direction shown in Fig. 4(a) changes from horizontal to the right at $\chi=0$ to vertically upward at $\chi=F$ for describing the negative dilatancy, and finally turns to the direction pointing to the left until χ reaches F_f for capturing the positive dilatancy. As for the plastic flow direction in the deviatoric plane, it is independent of the fractional-order μ , since no variable associated with the stress Lode angle contains in the yield function. That is, the plastic flow directions in the deviatoric plane for $\mu=0.52$ coincide with the plastic flow directions for $\mu=1$. Here, the coincident plastic flow directions are represented by the red arrows, as shown in Fig. 4(b). The plastic flow direction in the deviatoric plane is the radial direction along the current characteristic

stress Lode angle θ_β defined in Tab. A-1 (see Appendix A). The θ_β -values on the stress paths under the condition that $\theta=0^\circ$ and 60° are constant, while θ_β -values on the characteristic stress paths at $\theta=15^\circ$, 30° and 45° are changeable and increase with increasing shear stress. Therefore, the plastic flow directions along the radial direction change with θ_β -values accordingly, as shown in Fig. 4(b).

Fig. 3 Evolution of the non-orthogonal plastic flow direction with the expanding yield surface in the c_{ij} space.

Fig. 4 Yield curve and non-orthogonal plastic flow direction in (a) the meridian plane of the c_{ij} space and (b) the deviatoric plane of the c_{ij} space.

3.1.2 Plastic flow direction in the σ_{ij} space

The expansion of the ellipsoidal yield surface in the c_{ij} space corresponds to the evolution of the yield surface in the σ_{ij} space, as shown in Fig. 5. The plastic flow direction determined in the c_{ij}

space is applied to the corresponding stress state point on the yield surface in the σ_{ij} space. Corresponding to the expansion of the elliptic yield curve shown in Fig. 4(a), the evolution of the unsymmetrical yield curves in the meridian plane of the σ_{ij} space is shown in Fig. 6(a). As the shear stress increases along the stress path under triaxial compression conditions with a constant p , the plastic flow direction in Fig. 6(a) changes in the same way as that in Fig. 4(a). It changes from horizontal at stress ratio $\eta=0$ to vertically upward at $\eta=M$ and finally turns to the direction pointing to the left until η reaches M_f for capturing the dilatancy behaviour from negative to positive. In the deviatoric plane of the σ_{ij} space, the yield curve tends to change from an approximate circle to a curved triangle with the increase of η , and the plastic flow directions at different θ -values correspondingly change, as shown in Fig. 6(b). The plastic flow directions at $\theta=0^\circ$ and 60° coincide with the radial directions, while the plastic flow directions at $\theta=15^\circ$, 30° and 45° deviate leftward from the radial direction, and the degree of the deviation increases with the increase of η . This means the stress-induced anisotropy associated with the yield behaviour and plastic flow direction is reflected.

Fig. 5 Evolution of the yield surface and the non-orthogonal plastic flow direction in the σ_{ij} space.

Fig. 6 Yield curve and non-orthogonal plastic flow direction in (a) the meridian plane of the σ_{ij} space and (b) the deviatoric plane of the σ_{ij} space.

The above analysis illustrates that the non-orthogonal plastic flow rule [19] and the characteristic stress [40] are complementary in capturing soil behaviours under 3D stress conditions. The non-orthogonal plastic flow rule makes the plastic flow direction suitable for capturing the dilatancy of sand from negative to positive, while the characteristic stress extends the ability to capture dilatancy to 3D stress conditions. The combination of these two concepts gives rise to a new approach to capture the plastic flow direction for the sand under 3D stress conditions.

3.2 Hardening parameter for sand with dilatancy

Hardening parameter is an internal variable reflecting the hardening behaviours during the loading process. It is closely related to the determination of the plastic multiplier Λ in Eq. (7). In order to incorporate the dilatancy of sand from negative to positive into a non-orthogonal model, the hardening parameter should be formulated to be coordinated with the plastic flow direction. During the construction process, the hardening parameter should also be independent of the stress path [8, 29]. Obviously, the plastic volumetric strain as a hardening parameter is effective for normally consolidated clay. However, it is not suitable for describing initial negative and subsequent positive dilatancy of sand. By referring to the method proposed by Yao *et al.* [8], a stress-path-dependency factor Ω is introduced and combined with the volumetric strain to construct the hardening parameter H , which aims to make H stress-path-independent and capable of capturing the dilatancy of sand.

$$H = \int \Omega d\varepsilon_v^p \quad (15)$$

For normally consolidated clay, ε_v^p is used as the hardening parameter, so $\Omega=1$ is obtained.

The relationship $\varepsilon_v^p = l_p \ln c_{nx}/c_{n0}$ can be derived based on the linear relationship between e and $\ln(c_n)$ under isotropic compression conditions [19], where $l_p = (\lambda - \kappa)/[\beta \cdot (1 + e_0)]$, and λ is the compression index. Therefore, the equivalent characteristic consolidation pressure can be expressed as $c_{nx} = c_{n0} \exp(\varepsilon_v^p / l_p)$. For sand, H in Eq. (15) is adopted to replace ε_v^p , c_{nx} can thus be expressed as:

$$c_{nx} = c_{n0} \exp\left(\frac{H}{l_p}\right) = c_{n0} \exp\left[\frac{1}{l_p} \int \Omega d\varepsilon_v^p\right] \quad (16)$$

The partial derivative of the yield function f with respect to ε_v^p is required for determining Λ in Eq. (7) and can be obtained based on Eqs. (11) and (16):

$$\frac{\partial f}{\partial \varepsilon_v^p} = \frac{\partial f}{\partial c_{nx}} \frac{\partial c_{nx}}{\partial \varepsilon_v^p} = -\frac{\Omega}{l_p} N^2 c_n c_{nx} \quad (17)$$

In addition, the following differential equation is also needed and can be derived from Eq. (11):

$$\frac{\partial f}{\partial c_{ij}} dc_{ij} = N^2 (2c_n - c_{nx}) dc_n + 2c_s dc_s \quad (18)$$

By substituting Eqs. (17) and (18) together with Eq. (12) into Eq. (7), the specific expression of Λ is obtained:

$$\Lambda = \frac{l_p (N^2 c_n^2 - c_s^2) dc_n + 2c_s c_n dc_s}{\Omega c_n (c_s^2 + N^2 c_n^2) m_n} \quad (19)$$

By substituting Λ in Eq. (19) and (m_n, m_s) in Eq. (13) into Eq. (8), together with the stress ratio $\chi = c_n/c_s$, the plastic volumetric strain and the plastic shear strain can be expressed as:

$$\begin{cases} d\varepsilon_v^p = \Lambda \cdot m_n = \frac{l_p (N^2 - \chi^2) dc_n + 2\chi dc_s}{\Omega c_n (\chi^2 + N^2)} \\ d\varepsilon_s^p = \Lambda \cdot m_s = \frac{l_p (N^2 - \chi^2) dc_n + 2\chi dc_s}{\Omega c_n (\chi^2 + N^2)} \frac{2\chi^{2-\mu}}{\mu N^2 - (2-\mu)\chi^2} \end{cases} \quad (20)$$

The two stress paths shown in Fig. 7, include the constant mean characteristic principal stress path AB and the isotropic compression stress path AC , are used to derive the stress-path-dependent

254 factor Ω and an explicit expression of H .

256 **Fig. 7** Stress paths for deriving the hardening parameter H .

257 *3.2.1 Constant mean characteristic principal stress path AB ($dc_n=0$)*

258 Along the stress path AB , $dc_n=0$ and c_n is a constant. Therefore, $d\varepsilon_s^p$ in Eq. (20) can be
259 simplified as:

$$260 \quad d\varepsilon_s^p = \frac{1}{\Omega} \frac{4l_p \chi^{3-\mu} d\chi}{(\chi^2 + N^2) [\mu N^2 - (2-\mu)\chi^2]} \quad (21)$$

261 For the normally consolidated clay, $\Omega=1$. $d\varepsilon_s^p$ is expressed as:

$$262 \quad d\varepsilon_s^p = \frac{4l_p \chi^{3-\mu} d\chi}{(N^2 + \chi^2) [\mu N^2 - (2-\mu)\chi^2]} \quad (22)$$

263 Similar to Eq. (22), the expression of $d\varepsilon_s^p$ for sand can be assumed as:

$$264 \quad d\varepsilon_s^p = \rho \frac{4l_p \chi^{3-\mu} d\chi}{(N_f^2 + \chi^2) [\mu N_f^2 - (2-\mu)\chi^2]} \quad (23)$$

265 Here, ρ is an undetermined coefficient that reflects the ratio of the magnitude of $d\varepsilon_s^p$ between clay
266 and sand along the stress path AB . N in Eq. (22) is changed to N_f that related to the failure stress
267 ratio F_f of sand. In fact, F_f is not constant in describing the shearing process with some complex
268 behaviours, but for the sake of simplicity, it is assumed to be a constant. When sand reaches its
269 failure state at $\chi=F_f$, the expression of $d\varepsilon_s^p$ in Eq. (23) approaches infinity, which is similar to Eq.
270 (22) for clay. Thus, $N_f = F_f \sqrt{2/(\mu-1)}$ can be obtained.

271 Furthermore, by combining Eq. (21) with Eq. (23), the stress-path-dependent factor Ω is

272 derived and can be expressed as:

$$273 \quad \Omega = \frac{1}{\rho} \frac{(2-\mu)(F_f^2 - \chi^2)[(2-\mu)F_f^2 + \mu\chi^2]}{\mu(N^2 + \chi^2)[\mu N^2 - (2-\mu)\chi^2]} \quad (24)$$

274 Substituting Eq. (24) into Eq. (15) yields:

$$275 \quad H = \int \frac{1}{\rho} \frac{(2-\mu)(F_f^2 - \chi^2)[(2-\mu)F_f^2 + \mu\chi^2]}{\mu(N^2 + \chi^2)[\mu N^2 - (2-\mu)\chi^2]} d\varepsilon_v^p \quad (25)$$

276 3.2.2 Isotropic compression stress path AC ($dc_s=0$)

277 Under isotropic compression conditions ($dc_s=0$), χ is equal to 0. Eq. (25) can thus be simplified
278 as follows:

$$279 \quad H = \int \frac{1}{\rho} \frac{(2-\mu)^2 F_f^4}{\mu^2 N^4} d\varepsilon_v^p \quad (26)$$

280 Along stress path AC, only the plastic volumetric strain ε_v^p occurs for sand. That is, $H = \int d\varepsilon_v^p$
281 can be used to depict the hardening rule of sand under isotropic compression conditions. The
282 expression of ρ can thus be derived by combining $H = \int d\varepsilon_v^p$ with Eq. (26).

$$283 \quad \frac{1}{\rho} = \frac{\mu^2 N^4}{(2-\mu)^2 F_f^4} \quad (27)$$

284 By substituting Eq. (27) into Eq. (24), the stress-path-dependent factor Ω can be expressed as:

$$285 \quad \Omega = \frac{\mu N^4}{(2-\mu)F_f^4} \frac{(F_f^2 - \chi^2)[(2-\mu)F_f^2 + \mu\chi^2]}{(N^2 + \chi^2)[\mu N^2 - (2-\mu)\chi^2]} \quad (28)$$

286 Finally, the new hardening parameter is obtained by substituting Eq. (28) into Eq. (15) and can be
287 expressed as follows:

$$288 \quad H = \int \frac{\mu N^4}{(2-\mu)F_f^4} \frac{(F_f^2 - \chi^2)[(2-\mu)F_f^2 + \mu\chi^2]}{(N^2 + \chi^2)[\mu N^2 - (2-\mu)\chi^2]} d\varepsilon_v^p \quad (29)$$

289 With the proposed hardening parameter H , the negative and positive dilatancy behaviours of
290 sand can be described. Typical results of sand behaviours under triaxial compression conditions
291 under $p=\text{constant}$ are illustrated by the schematic diagram shown in Fig. 8(a), which is arranged in
292 terms of the stress ratio $\eta=q/p$ and the volumetric strain ε_v vs. the shear strain ε_s . Since $p=\text{constant}$,
293 $\varepsilon_{ij}^e = 0$ and $\varepsilon_{ij} = \varepsilon_{ij}^p$ are obtained. The negative and positive dilatancy can be explained and
294 indicated by the following equation derived from Eq. (29). Moreover, corresponding to Fig. 8(a),

the following conclusions listed in Tab. 1 can be drawn from Eq. (30) based on the non-negative dH .

$$d\varepsilon_v^p = \frac{(2-\mu)F_f^4}{\mu N^4} \frac{(N^2 + \chi^2) [\mu N^2 - (2-\mu)\chi^2]}{(F_f^2 - \chi^2) [(2-\mu)F_f^2 + \mu\chi^2]} dH \quad (30)$$

Tab. 1 Variation rule of $d\varepsilon_v^p$ with the increase of stress ratio η

Stress state	Characteristic stress state	dH	$d\varepsilon_v^p$	Current condition
$\eta=0$	$\chi=0$	$dH = d\varepsilon_v^p$	$d\varepsilon_v^p > 0$	Isotropic compression
$0 < \eta < M$	$0 < \chi < F$	$dH > 0$	$d\varepsilon_v^p > 0$	Negative dilatancy
$\eta=M$	$\chi=F$	$dH > 0$	$d\varepsilon_v^p = 0$	Phase transition state
$M < \eta \leq M_f$	$F < \chi \leq F_f$	$dH > 0$	$d\varepsilon_v^p < 0$	Positive dilatancy

By adopting H with the properties indicated above, the shear responses transiting from contraction to dilation are captured. When $M_f=M$, $F_f=F$ can be obtained, H will degenerate to ε_v^p , only negative dilatancy will be described, as shown in Fig. 8(b). For the condition that $\beta=1$, c_{ij} becomes σ_{ij} , $F=M$ and $F_f=M_f$ are obtained. When $N=M$, $\mu=1$ can be derived, so H can degenerate to the hardening parameter proposed by Yao *et al.* [8].

Fig. 8 Stress-strain relationship of soil with: (a) negative and positive dilatancy; and (b) negative dilatancy.

3.2.3 Stress-path-independency of the hardening parameter

The stress-path-independence of the hardening parameter H can be interpreted the change of H along different paths, as shown in Fig. 9 and Fig. 10. The yield curve passes through the stress state point $A_0(c_{nr0}, 0)$ which is the only intersection point of the yield surface f_0 and the c_n -axis. According to Eq. (16), the unique intersection point corresponds to a unique value of H_0 . That is, the yield surface f_0 is actually an equivalent surface of H_0 . As the loading continues along different paths to points B_1, C_1 and D_1 that are all on the yield surface f_1 as shown in Fig. 9(a), the yield surface gradually expands, and the intersection moves along the c_n -axis to the point $A_1(c_{nr1}, 0)$. Accordingly, the value of the hardening parameter changes to H_1 , which is equivalent for any stress state on f_1 . Therefore, the difference $\Delta H = H_1 - H_0$ can be calculated and is independent on the stress path. This is exactly the basic properties that hardening parameters need to satisfy. The corresponding yield surface and stress paths in the σ_{ij} space are shown in Fig. 9(b). Fig. 10 shows that, as the sand is loaded along different stress paths from f_0 to f_1 , the value of the H changes from H_0 to H_1 . The results of equal variation in H shown in Fig. 10 indicates that H is stress-path-independence.

Fig. 9 Stress paths for verification of the stress-path-independence of H : (a) in the characteristic stress space; (b) in the ordinary stress space.

Fig. 10 Stress-path-independence of H .

3.3 Stress-strain relationship

With the newly derived stress-path-dependent factor Ω , the explicit expression of the plastic multiplier Λ can be obtained based on Eq. (19):

$$\Lambda = l_p \frac{(2-\mu)F_f^4}{\mu N^4} \frac{(N^2 + \chi^2)[\mu N^2 - (2-\mu)\chi^2]}{(F_f^2 - \chi^2)[(2-\mu)F_f^2 + \mu\chi^2]} \frac{(N^2 c_n^2 - c_s^2)dc_n + 2c_s c_n dc_s}{c_n (c_s^2 + N^2 c_n^2) m_n} \quad (31)$$

By substituting Eqs. (14) and (31) into Eq. (2), the plastic strain increment is thus derived:

$$d\epsilon_{ij}^p = l_p \frac{(2-\mu)F_f^4 [\mu N^2 - (2-\mu)\chi^2] [(N^2 - \chi^2)dc_n + 2\chi dc_s]}{\mu N^4 c_n (F_f^2 - \chi^2) [(2-\mu)F_f^2 + \mu\chi^2]} \cdot \left[\frac{\delta_{ij}}{3} + \frac{3\chi^{1-\mu} (c_{ij}/c_n - \delta_{ij})}{\mu N^2 - (2-\mu)\chi^2} \right] \quad (32)$$

The expression of plastic strain increment is totally expressed by the characteristic stress, and thus naturally, the β criterion is used as the shear failure condition of sand. When $M_f=M$, $F_f=F$ can be obtained, and Eq. (32) can thus be simplified to determine the plastic strain increment for the loose sand with only negative dilatancy.

The total strain increment produced by the current stress increment can thus be obtained by Eq. (1), where the elastic strain increment for sand is expressed by Eq. (9) and the plastic strain increment is expressed by Eq. (32). The parameters involved in the non-orthogonal elastoplastic model can be easily determined, which will be discussed in the following section.

3.4 Parameter calibration

There are seven independent material parameters in the proposed model, i.e., λ , κ , ν , φ , φ_f , β and ζ . All material parameters have clear physical meaning and can be easily determined by conventional triaxial tests. The parameters λ , κ and ν are the same with the parameters of the

modified Cam-clay model. The parameters φ and φ_f are the internal friction angles of the sand under triaxial compression conditions at the phase transition state and the failure state, respectively. β is the characteristic stress index, which can be determined based on the expression of the characteristic failure stress ratio F_f listed in Tab. A-1 (see Appendix A). Actually, F_f can be determined by combining the internal friction angle φ_b at any intermediate principal stress coefficient b ($b=(\sigma_2-\sigma_3)/(\sigma_1-\sigma_3)$) with β based on the expression of the β criterion in Eq. (39). Note the value of F_f is independent of b , i.e., $F_f|_{b_1} = F_f|_{b_2}$, where the subscripts b_1 and b_2 are two different b -values. Based on this, the parameter β can be determined by φ_{b_1} and φ_{b_2} , which reflects the relative proportional relationship of shear strength at different b . In particular, φ_f at $b_1=0$ and φ_{fe} at $b_2=1$ are used to determine β -value based on the method proposed by Lu *et al.* [19,40]. Parameter ξ is called the spacing ratio [21,30]. It can also be used to describe the change of the void ratio $\Delta\Pi$ of sand between the initial state and the phase transition state under triaxial compression conditions under $p=\text{const}$. The change of the void ratio has been derived and expressed as $\Delta\Pi=(\lambda-\kappa)\cdot\ln(p_x/p_{x0})$ [21], where p_{x0} and p_x are the equivalent mean principal stresses for the initial state and the phase transition state, respectively. Here, $p_x/p_{x0}=\xi$ is assumed, therefore, ξ can be determined by $\xi=\exp[\Delta\Pi/(\lambda-\kappa)]$, and $\xi=2$ is a special case derived from the modified Cam-clay model.

In addition, N and μ are two model parameters, which can both be easily determined based on the above material parameters. Their determination is given below.

Since the equivalent mean principal stresses p_{x0} and p_x are both defined for the isotropic compression conditions, the equivalent mean characteristic principal stresses can be expressed as $c_{nx0}=p_r\cdot(p_{x0}/p_r)^\beta$ and $c_{nx}=p_r\cdot(p_x/p_r)^\beta$. Therefore, based on $\xi=p_x/p_{x0}$ mentioned above, we can get $c_{nx}=\xi^\beta\cdot c_{nx0}$. On the other hand, substituting the phase transition stress state $(c_{nt}, c_{st})=(c_{nt}, F\cdot c_{nt})$ into Eq. (12) yields:

$$c_{nx} = c_{nt} \left(\frac{F^2}{N^2} + 1 \right) \quad (33)$$

By combining $c_{nx}=\xi^\beta\cdot c_{nx0}$ with Eq. (33), N can be derived as:

$$N = F \sqrt{\frac{c_{nt}}{\xi^\beta c_{nx0} - c_{nt}}} \quad (34)$$

where $c_{nt}=(c_{1t}+2c_{3t})/3$, in which $c_{1t}=p_r\cdot(\sigma_{1t}/p_r)^\beta$ and $c_{3t}=p_r\cdot(\sigma_{3t}/p_r)^\beta$ can be obtained from Tab. A-1. σ_{1t} and σ_{3t} are the major and minor principal stresses at the phase transition state, which can be

calculated from $\sigma_{1t}=p_{x0}(1+2M/3)$ and $\sigma_{3t}=p_{x0}(1-M/3)$, respectively. Therefore, parameter N can be determined by the following equation:

$$N = F \sqrt{\frac{(1+2M/3)^\beta + 2(1-M/3)^\beta}{3\xi^\beta - (1+2M/3)^\beta - 2(1-M/3)^\beta}} \quad (35)$$

The shape of the yield surface is known when N is determined. Therefore, the expression of plastic flow direction is obtained, and the stress-dilatancy ratio d can be correspondingly derived:

$$d = \frac{d\varepsilon_v^p}{d\varepsilon_s^p} = \frac{\mu N^2 - (2 - \mu)\chi^2}{2\chi^{2-\mu}} \quad (36)$$

When the phase transition state reaches, χ reaches F , and d in Eq. (36) is equal to 0, then the fractional-order μ can be determined by the following equation:

$$\mu = \frac{2F^2}{N^2 + F^2} \quad (37)$$

4 Model verification

The performance of the proposed model can be evaluated and verified by comparing model prediction with test results of sand from literature. We consider the tests on saturated Toyoura sand under conventional triaxial compression ($\sigma_3=p_0$, $p=p_0$ and $\sigma_1=p_0$) and extension ($\sigma_3=p_0$, $p=p_0$ and $\sigma_1=p_0$) conditions and true triaxial conditions ($\theta=15^\circ$, 30° and 45°) [29]. The mean diameter of sand is 0.2mm, the uniformity coefficient is 1.3 and the specimens were prepared in medium dense with an initial void ratio 0.68. The material parameters used for model prediction are listed in Tab. 2. The corresponding model parameters $N=0.439$ and $\mu=0.390$ can thus be determined based on the material parameters by the method mentioned in Section 3.4.

Tab. 2 Material parameters used for model prediction.

$\lambda/(1+e_0)$	$\kappa/(1+e_0)$	ν	φ	φ_f	β	ξ
0.00403	0.00251	0.3	24.209°	40.475°	0.24	2.3

4.1.1 Toyoura sand under conventional triaxial conditions

Test data of saturated Toyoura sand under conventional triaxial compression ($\sigma_3=p_0$, $p=p_0$ and $\sigma_1=p_0$) and extension ($\sigma_3=p_0$, $p=p_0$ and $\sigma_1=p_0$) conditions are digitized from [8] and indicated by open circles, as shown in Fig. 11. As shown in Fig. 11(a) and (b), the negative dilatancy of sand is obvious, since p increases under triaxial compression and extension conditions when $\sigma_3=p_0$. Less negative dilatancy can be found in Fig. 11(c) and (d), since a constant p is applied. Moreover, due to the

decrease of p under triaxial compression and extension conditions when $\sigma_1=p_0$, the least negative dilatancy is found in Fig. 11(e) and (f). The test results of Toyoura sand along six different stress paths are predicted, as shown in Fig. 11(a)-(f). Due to the contributions of the non-orthogonal plastic flow direction and the newly constructed H , different degrees of dilatancy from negative to positive along six different stress paths are well predicted. Comparisons between the test results and the prediction show that the relationship of ε_1 vs. σ_1/σ_3 and ε_3 vs. σ_1/σ_3 is also well captured by the proposed model. On the other hand, benefit from the application of characteristic stress, the β criterion is naturally used in the model, and thus the tendency that the shear strength of the sand under extension conditions is greater than that under compression conditions can be well reflected.

(a) Compression when $\sigma_3=p_0$

(b) Extension when $\sigma_3=p_0$

(c) Compression when $p=p_0$

(d) Extension when $p=p_0$

(e) Compression when $\sigma_1 = p_0$

(f) Extension when $\sigma_1 = p_0$

Fig. 11 Comparison between predicted and test results under conventional triaxial conditions.

4.1.2 Toyoura sand under true triaxial conditions

True triaxial drained tests under constant mean principal stress $p = 196 \text{ kPa}$ [29] can be used to demonstrate the capability of the 3D elastoplastic models for sand [29, 41]. The values of θ are 15° , 30° and 45° , respectively. Triaxial tests under $p = 196$ when $\theta = 0^\circ$ (triaxial compression) and 60° (triaxial extension) have been used to verify the proposed non-orthogonal model, as shown in Fig. 11(c) and (d). The capability of the proposed model in capturing the behaviour of sand under 3D stress conditions can be verified by the predicting test results of sand, as shown in Fig. 12(a)-(c). The comparisons are arranged in terms of the relation among the principal stress ratio σ_1/σ_3 , volumetric strain ε_v , and principal strains ε_i ($i=1,2,3$). In addition to well capture the volume change from negative to positive under 3D stress conditions, it is also critical for a 3D model for sand to reflect three-dimensional strength behaviour and the three-dimensional plastic flow direction. For the strength property of sand under 3D stress conditions ($\theta = 0^\circ, 15^\circ, 35^\circ, 45^\circ$ and 60°), predictions are in good agreement with the experiment results. Moreover, an appropriate description of the plastic flow direction in the deviatoric plane, as illustrated in Section 3.1, provides the proposed model with the ability to capture the relative relationship between different principal strains, as shown in Fig. 12(a)-(c).

(a) $\theta=15^\circ$

(b) $\theta=30^\circ$

(c) $\theta=45^\circ$

Fig. 12 Comparison between predicted and test results under true triaxial conditions when $p=p_0$.

5 Conclusions

In this paper, a non-orthogonal elastoplastic model was proposed within the framework of the non-orthogonal elastoplastic model. Dilatancy behaviours of sand from initial negative to subsequent positive along different stress paths were well described. Only seven material parameters with clear physical meanings are needed and can be easily determined by conventional triaxial tests.

Model performance and comparison with test results of sand show that the proposed model is capable of capturing the salient behaviour of sand under 3D stress conditions. The key characteristics are listed as follows:

1. The framework of the 3D non-orthogonal elastoplastic model was established by introducing the characteristic stress, which makes the developed model be born with the potential to capture 3D mechanical behaviour of soil.
2. The plastic flow direction for describing the dilatancy of sand was determined by the non-orthogonal plastic flow rule. Moreover, the complementarity between the non-orthogonal plastic flow rule and the characteristic stress in capturing the 3D plastic flow direction of sand was illustrated.
3. A hardening parameter was newly constructed for capturing the negative and positive dilatancy of sand during the shearing process, which is coordinated with the non-orthogonal plastic flow direction. In addition, its independency from the stress path was demonstrated.

Acknowledgements

This study was supported by the National Key Research and Development Program of China (Grant No. 2018YFC1504301), National Natural Science Foundation of China (Grant Nos. 51778026, 51421005 and U1839201), and the China Scholar Council. The European Commission for the support to the projects FRAMED “Fracture Across Scales and Materials, Processes and Disciplines” (Grant No. 734485), and HERCULES “Towards geohazards resilient infrastructure under changing climates” (Grant No. 778360) are gratefully acknowledged.

Appendix A Characteristic stress

The characteristic stress concept was proposed to establish a unified strength criterion (simplified as β criterion) in a simple way ^[40], and has been applied to develop 3D elastoplastic model for soil ^[19]. The expression of the characteristic stress tensor can be written as:

$$c_{ij} = p_r \left[\left(\frac{\sigma_1}{p_r} \right)^\beta n_i^{(1)} n_j^{(1)} + \left(\frac{\sigma_2}{p_r} \right)^\beta n_i^{(2)} n_j^{(2)} + \left(\frac{\sigma_3}{p_r} \right)^\beta n_i^{(3)} n_j^{(3)} \right] \quad (38)$$

where c_{ij} is the characteristic stress tensor whose principal directions coincide with those of the ordinary stress tensor σ_{ij} . And the expression of the characteristic principal stress $c_k = p_r (\sigma_k / p_r)^\beta$ can be obtained the based on Eq. (38). σ_k ($k=1, 2, 3$) is the principal stress. p_r is the reference stress

471 for dimensionless transformation, and $p_r=1\text{kPa}$ is commonly used. β is the characteristic stress index,
 472 which is a non-dimensional material parameter that ranges from 0 to 1. $n_i^{(k)}$ in Eq. (38) is the
 473 corresponding directional cosine of the direction vector $e^{(k)}$ of the principal stress, that is, the
 474 principal stress vector can be expressed as $c_k e^{(k)}$. Therefore, the spectral decomposition of the
 475 characteristic stress tensor can be illustrated by the schematic diagram shown in Fig. 13.

476
477 Fig. 13 Spectral decomposition of the characteristic stress tensor.

478 Based on the friction rule on the octahedral plane under the action of the c_{ij} , a nonlinear unified
 479 β strength criterion [40] was developed, which is used as the failure criterion in this paper:

480
$$\frac{c_s}{c_n} = F_f \quad (39)$$

481 where F_f is the characteristic failure stress ratio. The specific expressions of the characteristic stress
 482 variables and parameters are listed in Tab. A-1, which are also compared with the ordinary stress.

483 Tab. A-1 Comparison between the ordinary stress and the characteristic stress.

	Ordinary stress	Characteristic stress
Stress tensor	σ_{ij}	c_{ij}
Principal stress	$\sigma_i, \quad i=1,2,3$	$c_i = p_r \left(\frac{\sigma_i}{p_r} \right)^\beta, \quad i=1,2,3$
Mean stress	$p = \sigma_{ij} \delta_{ij} / 3$	$c_n = c_{ij} \delta_{ij} / 3$
General shear stress	$q = \sqrt{\frac{3}{2} (\sigma_{ij} - p \delta_{ij}) (\sigma_{ij} - p \delta_{ij})}$	$c_s = \sqrt{\frac{3}{2} (c_{ij} - c_n \delta_{ij}) (c_{ij} - c_n \delta_{ij})}$
Stress Lode angle	$\theta = \arctan \left[\frac{\sqrt{3} (\sigma_2 - \sigma_3)}{2\sigma_1 - \sigma_2 - \sigma_3} \right]$	$\theta_\beta = \arctan \left[\frac{\sqrt{3} (c_2 - c_3)}{2c_1 - c_2 - c_3} \right]$

Stress ratio	$\eta=q/p$	$\chi=c_s/c_n$
Phase transition stress ratio	$M = \frac{6 \sin \varphi}{3 - \sin \varphi}$	$F = 3 \frac{(1 + \sin \varphi)^\beta - (1 - \sin \varphi)^\beta}{(1 + \sin \varphi)^\beta + 2(1 - \sin \varphi)^\beta}$
Failure stress ratio	$M_f = \frac{6 \sin \varphi_f}{3 - \sin \varphi_f}$	$F_f = 3 \frac{(1 + \sin \varphi_f)^\beta - (1 - \sin \varphi_f)^\beta}{(1 + \sin \varphi_f)^\beta + 2(1 - \sin \varphi_f)^\beta}$

Appendix B Specific expressions of fractional derivative

The specific expressions of the Caputo fractional derivative of a power function x^n is given and expressed as:

$${}_0D_x^\mu(x^n) = \frac{\partial^\mu(x^n)}{\partial x^\mu} = \frac{\Gamma(n+1)x^{n-\mu}}{\Gamma(n+1-\mu)} \quad (40)$$

where ${}_0D_x^\mu$ means the Caputo derivative of order μ . 0 is chosen as the starting point and x is the terminal point in the operator. $\Gamma(\cdot)$ indicates the Gamma function and its expression is $\Gamma(z) = \int_0^\infty e^{-\tau} \tau^{z-1} d\tau$, which satisfies the recursive relation, i.e., $\Gamma(x+1) = x \cdot \Gamma(x)$. In addition, it is worth to note that ${}_0D_x^\mu(C) = 0$, where C is a constant. The linear relationship of the Caputo fractional derivatives is also necessary and can be expressed as:

$${}_0D_x^\mu[A_1 f_1(x) + A_2 f_2(x)] = A_1 \cdot {}_0D_x^\mu f_1(x) + A_2 \cdot {}_0D_x^\mu f_2(x) \quad (41)$$

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