Non-orthogonal elastoplastic constitutive model with the critical state for clay

Jingyu LIANG¹, Dechun LU^{2*}, Xin ZHOU³, Xiuli DU⁴, Wei WU⁵

ABSTRACT:

A non-orthogonal elastoplastic model for clay is proposed by combining the non-orthogonal plastic flow rule with the critical state concept and presented from the perspective of the magnitude and direction of the plastic strain increment. The magnitude is obtained based on the improved elliptical yield function and the plastic volumetric strain dependent hardening parameter. The direction is determined by applying the non-orthogonal plastic flow rule with the Riemann-Liouville fractional derivative to the yield function without the necessity of additional plastic potential function. The presented approach gives rise to a simple model with five parameters. All parameters have clear physical meaning and can be easily identified by triaxial tests. The model performance is shown by analyzing the evolution process of the yield surface, the hardening rule and the plastic flow direction. The capability of the proposed model to capture the mechanical behaviours of clay with different stiffness is also confirmed by predicting test results from the literature.

Keywords: Clay; Constitutive model; Elastoplasticity; Non-orthogonal plastic flow rule; Stress dilatancy; Critical state

¹ Ph.D. Candidate. Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China. E-mail: liangjingyu@emails.bjut.edu.cn.

^{2*} Professor. Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China (Corresponding author). E-mail: dechun@bjut.edu.cn.

³ Ph.D. Candidate. Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China. E-mail: zhouxin@emails.bjut.edu.cn.

⁴ Professor. Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China. E-mail: duxuli@bjut.edu.cn.

⁵ Professor. Institute of Geotechnical Engineering, University of Natural Resources and Life Sciences, Vienna, 1180, Austria. E-mail: wei.wu@boku.ac.at.

1 Introduction

The constitutive model can be used to determine the two basic elements of the strain increment produced by the current stress increment, namely (i) magnitude and (ii) direction. Within the framework of plasticity theory, the magnitude of the plastic strain increment can be determined by the consistency condition together with the yield function and the hardening parameter. The direction of the plastic strain increment, i.e., the plastic flow direction, is usually determined by the orthogonal gradient of the plastic potential surface. If the plastic potential function is chosen to be the same with the yield function, the associated flow rule is adopted [1-4], in which the plastic flow direction is orthogonal to the yield surface. But the associated flow rule has been reported to be unsuitable for describing the mechanical behaviours of geomaterials [5-7]. If the plastic potential function is not identical to the yield function, the non-associated flow rule is adopted [8-11], and the plastic flow direction perpendicular to the plastic potential surface is non-orthogonal to the yield surface. It is also called the non-orthogonal conditions, but the non-orthogonality with respect to the yield surface is accomplished by constructing a new plastic potential function [12-15]. However, it is difficult to construct a required plastic potential function, especially for geomaterials with pressure sensitivity and dilatancy [16-19]. In addition, some additional parameters without clear physical meanings might be needed.

The direction of plastic strain increment can also be directly determined by the non-orthogonal gradient of the yield function. Therefore, the plastic potential function is no longer necessary for determining the plastic flow direction. The establishing process of the elastoplastic model becomes simpler and few material parameters are required. The fractional partial derivative possesses the ability to adjust the non-orthogonal gradient direction of the differentiable function [20, 21]. In order to determine the general tensor form of the plastic strain increment based on the commonly used yield function for soil, a general form of non-orthogonal plastic flow rule [22, 23] is proposed based on the fractional partial derivative and the covariant transformation. The proposed non-orthogonal plastic flow rule has been applied to establish elastoplastic models for soil [22] and concrete [23]. Nevertheless, the non-orthogonal plastic flow rule has not been directly introduced into the critical state theory. Their combination is promising for reproducing the mechanical behaviour of soil. During establishing non-orthogonal elastoplastic models for soil [21, 22], the plastic flow direction is expressed by the Caputo fractional derivative. However, the Caputo fractional derivative of constant

is equal to zero, which makes it unsuitable for the elliptical yield function of the modified Cam-clay model when the fractional order is greater than one.

In this paper, a framework of non-orthogonal elastoplastic critical state model for clay is presented. A general non-orthogonal plastic flow rule with the Riemann-Liouville (RL) fractional derivative is developed to determine the plastic flow direction without recourse to the plastic potential function. The proposed model is capable of reasonably capturing the mechanical behaviour of clay with different stiffness.

2 Stress-strain relationship of the non-orthogonal model for clay

In what follows, the main ingredients of the non-orthogonal elastoplastic model are recapitulated. The strain increment is decomposed into two components: the reversible elastic strain increment $d\epsilon_{ij}^e$ and the irreversible plastic strain increment $d\epsilon_{ij}^p$:

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \quad (1)$$

where $d\epsilon_{ij}^e$ can be determined by the Hooke's law and the $d\epsilon_{ij}^p$ can be determined based on the non-orthogonal plastic flow rule in this paper.

2.1 Elastic strain increment

The elastic strain increment can be obtained from the Hooke's law, in which the principal axis direction of the elastic strain increment is coincide with that of stress increment and the magnitude of the elastic strain increment can be determined by the elastic parameters as follows:

$$d\epsilon_{ij}^e = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} \quad (2)$$

where ν is the Poisson's ratio. E is the Young's modulus, which can be determined from the linear relationship between void ratio e and the natural logarithm of mean stress p , i.e., e - $\ln p$ relationship, as shown in Fig. 1. The expression of E in Hooke's law is:

$$E = 3(1-2\nu) \frac{1+e_0}{\kappa} p \quad (3)$$

where e_0 is the initial void ratio. κ is the swelling index of the swelling line (SWL: $e=\Pi_S-\kappa \cdot \ln p$, in which Π_S is the void ratio at $p=1$ kPa on the SWL), as shown in Fig. 1. The symbol λ in Fig. 1 is the compression index of the normally consolidated line (NCL: $e=\Pi_N-\lambda \cdot \ln p$, where Π_N is the void ratio at $p=1$ kPa on the NCL). The determined method of E is the same as that in the critical state theory [1]. Apparently, the modulus E in Eq. (3) is variable and depends on p , which reflects the pressure

89 dependence of clay.

Fig. 1 e - $\ln p$ relationship under isotropic consolidation conditions.

2.2 Plastic strain increment

The plastic strain increment can be expressed as:

$$d\varepsilon_{ij}^p = \Lambda \cdot m_{ij} \quad (4)$$

where Λ is the plastic multiplier that represents the magnitude of $d\varepsilon_{ij}^p$. Λ can be determined by submitting the yield function and the hardening parameter into the consistency condition. m_{ij} represents the direction of $d\varepsilon_{ij}^p$, which can be determined by the non-orthogonal plastic flow rule.

2.2.1 Magnitude of plastic strain increment

Within the framework of plasticity theory, yield function and hardening parameter are the main parts to determine the magnitude of plastic strain increment. Yield surface is the equivalent surface of the hardening parameter, and its general form is:

$$f = f\left[\sigma_{ij}, H\left(\varepsilon_{kl}^p\right)\right] \quad (5)$$

where H is the hardening parameter, which is the function of specific plastic strain.

The consistency condition ensures the consistency of the stress state and the hardening parameter, which makes the yield function with the hardening parameter value always pass through the current stress state point. It is used to close the set of constitutive equations and determine the plastic multiplier Λ . The consistency condition states that $df=0$ at yield, which is the total differential of the yield function f in Eq. (5). Therefore, for the isotropic hardening rule, the consistency condition can be expressed as:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial H} \frac{\partial H}{\partial \varepsilon_{kl}^p} d\varepsilon_{kl}^p = 0 \quad (6)$$

By submitting Eq. (4) into Eq. (6), the plastic multiplier Λ can be obtained:

$$\Lambda = - \frac{\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\frac{\partial f}{\partial H} \frac{\partial H}{\partial \varepsilon_{kl}^p} m_{kl}} \quad (7)$$

Once the yield function and the hardening parameter is decided, the expression of Λ can be accordingly obtained.

In this paper, an improved elliptical yield surface, which has the same form and convexity with the MCC yield function [1], is adopted and expressed as:

$$f = q^2 + N^2(p^2 - p_x p) = 0 \quad (8)$$

where p_x is the equivalent consolidation stress, which is the intersection coordinate of the yield curve and the p -axis and can be expressed by any stress state (p, q) on the yield surface, i.e.,

$p_x = q^2 / N^2 p + p$. N is the axial ratio of the vertical and horizontal axes of the elliptic yield curve.

If N is forced to be the same with the critical state stress ratio M , Eq. (8) becomes the yield function of the MCC model. If N does not equal M , there will be two possible scenarios for the improved yield surface based on the relative relationship between N and M : (a) $N < M$; (b) $N' > M$, where N' is used as the symbol for the condition $N > M$. The corresponding shapes of yield curve are illustrated in Fig. 2.

Fig. 2 Possible shapes of the yield curve when: (a) $N < M$; (b) $N' > M$.

An experimentally validated form for the isotropic hardening parameter, which is the same with the MCC model [1], is used in this paper to reflect the strain hardening feature of clay. Based

on the e - $\ln p$ relationship under isotropic compression conditions as shown in Fig. 1, the hardening parameter H can be expressed as:

$$H = \varepsilon_v^p = l_p \cdot \ln \frac{p_x}{p_0} \quad (9)$$

where $l_p = (\lambda - \kappa) / (1 + e_0)$. p_x connects the hardening parameter ε_v^p with the yield function, and its specific expression associated with ε_v^p can be derived from Eq. (9):

$$p_x = p_0 \exp \left(\frac{\varepsilon_v^p}{l_p} \right) \quad (10)$$

With the expressions of the yield function in Eq. (8) and the volumetric strain dependent hardening parameter in Eq. (9) together with its incremental expression $d\varepsilon_v^p = \Lambda \cdot m_p$ from Eq. (4), the expression of Λ in Eq. (7) can be converted accordingly as follows:

$$\Lambda = - \frac{\frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial q} dq}{\frac{\partial f}{\partial \varepsilon_v^p} m_p} \quad (11)$$

The partial derivatives of the yield function f in Eq. (11) can be derived from Eqs. (8) and (9). Therefore, Λ can be expressed as:

$$\Lambda = l_p \frac{N^2 (2p - p_x) dp + 2q dq}{N^2 p p_x m_p} \quad (12)$$

By substituting $p_x = q^2 / (N^2 p) + p$ from Eq. (8) into Eq. (12),

$$\Lambda = l_p \frac{(N^2 p^2 - q^2) dp + 2p q dq}{p (q^2 + N^2 p^2) m_p} \quad (13)$$

Particularly, as indicated by Eq. (13), m_p associated with the volumetric strain dependent hardening parameter affects the calculation of Λ , which will be discussed in the below section.

2.2.2 Direction of plastic strain increment

The direction of plastic strain increment is determined by the non-orthogonal plastic flow rule, which is the non-orthogonal gradient of the yield surface. By combining the fractional derivative and the covariant transformation, Lu et al. [22, 23] proposed a non-orthogonal plastic flow rule in general form, which can be expressed as:

$$d\varepsilon_{ij}^p = \Lambda \cdot m_{ij} = \Lambda \cdot \frac{\partial^\mu f}{\partial S_{kl}^\mu} \frac{\partial S_{kl}}{\partial \sigma_{ij}} \quad (14)$$

where S_{kl} is the variable that constructs the yield function, which can be the stress invariants of σ_{ij} , like the commonly used (p, q, θ) . The plastic flow direction m_{ij} is actually the non-orthogonal gradient of the yield surface, which is determined by the fractional order partial derivative $\partial^\mu f / \partial S_{kl}^\mu$ in S_{kl} space firstly and then transformed into σ_{ij} space by the covariant transformation $\partial S_{kl} / \partial \sigma_{ij}$. Actually, the general tensor form of m_{ij} in Eq. (14) is the kernel for deriving the general expression of $d\varepsilon_{ij}^p$ that is essential for model application and the numerical research.

The geometric schematic diagram of the non-orthogonal plastic flow rule expressed based on the fractional derivative is shown in Fig. 3. The components (m_1, m_2, m_3) of the plastic strain increment direction can be determined by the non-orthogonal plastic flow rule. The resultant direction vector $\mathbf{m}$ indicated by the red arrow in Fig. 3(a) is the fractional order gradient of the yield function, which is not orthogonal to the yield surface. By contrast, the vector $\mathbf{n}$ indicated by the black arrow is the gradient of the yield surface, i.e., $\mathbf{m}$ when $\mu=1$, which is orthogonal to the yield surface, as shown in Fig. 3(b).

Fig. 3 Comparison of the fractional gradient vector when $\mu \neq 1$ and the normal vector when $\mu = 1$.

The RL operator of the fractional partial derivative is used in this paper to derive the explicit solution of the fractional derivative of Eq. (8). The specific expressions of the RL fractional derivative for a power function and a constant as well as the linear relation of the fractional derivative are required and listed as follows:

$${}_0D_x^\mu(x^n) = \frac{\partial^\mu(x^n)}{\partial x^\mu} = \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} x^{n-\mu} \quad (15)$$

$${}^0D_x^\mu(C) = \frac{\partial^\mu(C)}{\partial x^\mu} = \frac{Cx^{-\mu}}{\Gamma(1-\mu)} \quad (16)$$

$${}^0D_x^\mu[A_1f_1(x) + A_2f_2(x)] = A_1 \cdot {}^0D_x^\mu f_1(x) + A_2 \cdot {}^0D_x^\mu f_2(x) \quad (17)$$

where ${}^0D_x^\mu$ is the operator of the fractional derivative of order μ , and 0 is chosen as the starting point of the fractional derivative. $\Gamma(z) = \int_0^\infty e^{-\tau} \tau^{z-1} d\tau$, ($\text{Re}(z) > 1$) is the Euler Gamma function.

Based on Eq. (14), the non-orthogonal plastic flow direction m_{ij} can be expressed as:

$$m_{ij} = \frac{\partial^\mu f}{\partial p^\mu} \frac{\partial p}{\partial \sigma_{ij}} + \frac{\partial^\mu f}{\partial q^\mu} \frac{\partial q}{\partial \sigma_{ij}} \quad (18)$$

where the fractional partial derivative of the yield function in Eq. (8) with respect to p and q can be expressed as $(m_p, m_q) = (\partial^\mu f / \partial p^\mu, \partial^\mu f / \partial q^\mu)$, which is determined by applying the RL operator to Eq. (8). Based on Eqs. (15)-(17), $p_x = q^2 / (N^2 p) + p$ from Eq. (8) and the recursive property of the gamma function, i.e., $\Gamma(x+1) = x \cdot \Gamma(x)$, (m_p, m_q) can be thus obtained:

$$(m_p, m_q) = \left[\frac{(\mu^2 - 2\mu)q^2 + \mu N^2 p^2}{\Gamma(3-\mu)p^\mu}, \frac{(3\mu - \mu^2)q^{2-\mu}}{\Gamma(3-\mu)} \right] \quad (19)$$

The covariant transformation coefficient between (p, q) and the ordinary stress tensor σ_{ij} can be expressed as:

$$\begin{cases} \frac{\partial p}{\partial \sigma_{ij}} = \frac{1}{3} \delta_{ij} \\ \frac{\partial q}{\partial \sigma_{ij}} = \frac{3(\sigma_{ij} - p\delta_{ij})}{2q} \end{cases} \quad (20)$$

By substituting Eqs. (19) and (20) into Eq. (18), the explicit expression of the plastic strain increment direction is expressed into the tensor form:

$$m_{ij} = \frac{(\mu^2 - 2\mu)q^2 + \mu N^2 p^2}{\Gamma(3-\mu)p^\mu} \frac{1}{3} \delta_{ij} + \frac{(3\mu - \mu^2)q^{2-\mu}}{\Gamma(3-\mu)} \frac{3(\sigma_{ij} - p\delta_{ij})}{2q} \quad (21)$$

By submitting Eq. (13) and Eq. (21) into Eq. (4), the plastic strain increment is obtained:

$$d\varepsilon_{ij}^p = \frac{\lambda - \kappa}{1 + e_0} \frac{(N^2 p^2 - q^2) dp + 2pq dq}{p(q^2 + N^2 p^2)} \left[\frac{1}{3} \delta_{ij} + \frac{3(3-\mu)p^\mu q^{1-\mu}(\sigma_{ij} - p\delta_{ij})}{2N^2 p^2 - 2(2-\mu)q^2} \right] \quad (22)$$

Specifically, when $\mu=1$ and $N=M$, Eq. (22) can also degenerate into the explicit expression of the MCC model:

$$d\varepsilon_{ij}^p = \frac{\lambda - \kappa}{1 + e_0} \frac{(M^2 p^2 - q^2) dp + 2pq dq}{p(q^2 + M^2 p^2)} \left[\frac{\delta_{ij}}{3} + \frac{3p(\sigma_{ij} - p\delta_{ij})}{M^2 p^2 - q^2} \right] \quad (23)$$

Finally, the stress-strain relationship for the non-orthogonal critical state elastoplastic constitutive model is obtained by submitting Eqs. (2) and (22) into Eq. (1).

2.3 Parameter calibration

There are only five material parameters with clear physical meaning in the proposed non-orthogonal critical state model, among which four material parameters (λ , κ , M and ν) are identical to the MCC model. The critical state stress ratio M can be represented by the internal friction angel φ under triaxial compression conditions:

$$M = \frac{6 \sin \varphi}{3 - \sin \varphi} \quad (24)$$

Only the critical void ratio change $\Delta\Pi = \Pi_N - \Pi_C$ is added, which is the vertical distance between the NCL and the CSL, as shown in Fig. 4. Under triaxial compression conditions from A to B shown in Fig. 4, the change of the void ratio Δe^p is exactly equal to $\Delta\Pi$. It also equals the void ratio variation along the path AA_xB , as shown in Fig. 4. The void ratio variations corresponding to the path AA_x and the path A_xB are Δe and Δe^e , respectively. And $\Delta e = \lambda \cdot \ln(p_x/p_0)$ and $\Delta e^e = \kappa \cdot \ln(p_x/p_0)$ are easily obtained based on the expressions of the NCL and the SWL. Therefore, $\Delta e^p = \Delta e - \Delta e^e$ can be expressed as $\Delta e^p = \Delta\Pi = (\lambda - \kappa) \cdot \ln \xi$, where $\xi = p_x/p_0$. Once the parameters λ and κ are determined, ξ and $\Delta\Pi$ are one-to-one corresponding parameters.

$$\xi = \exp\left(\frac{\Delta\Pi}{\lambda - \kappa}\right) \quad (25)$$

Fig. 4 Vertical distance between the NCL and the CSL.

The fractional order μ and the axial ratio N of the yield curve are two model parameters used

in this paper, which are not independent and can be determined by ξ . According to the consistency condition, the yield curve finally passes through the critical state stress point $(p_{cs}, q_{cs}) = (p_0, M \cdot p_0)$. By submitting $(p_0, M \cdot p_0)$ into Eq. (8) and together with $\xi = p_x/p_0$, N can be determined:

$$N = \frac{M}{\sqrt{\xi} - 1} \quad (26)$$

In particular, $\xi=2$ can be determined from the expressions of the MCC yield function, and $\Delta\Pi = (\lambda - \kappa) \cdot \ln 2$ is settled accordingly. The fractional order μ can be determined based on the stress-dilatancy ratio d equals 0 at the critical stress state. The expression of d is presented in details in section 3.2. And thus, by combining Eq. (26) with $d=0$ in Eq. (28), one can get μ as follows:

$$\mu = 2 - \frac{1}{\xi - 1} \quad (27)$$

From Eqs. (26) and (27), we can see that μ and N are not independent but coordinated with the critical state concept. Only one parameter ξ or $\Delta\Pi$ is needed. When $\xi > 2$, $\mu > 1$ and $N < M$ can be obtained, which reflects that the clay is soft with the stiffness lower than that of the MCC model. When $\xi < 2$, $\mu < 1$ and $N > M$ can be obtained, which reflects that the clay is hard with the stiffness greater than that of the MCC model. Compared with the MCC model, by adding only one parameter ξ (or $\Delta\Pi$), a non-orthogonal critical state model can be easily developed and can be used to describe behaviours of clay with different stiffness.

3 Performance of the model

The performance of the non-orthogonal elastoplastic model can be visualized by analyzing the evolution of the yield surface, the hardening rule and plastic flow direction. By matching and coordinating the yield curve shape, the hardening parameter and the non-orthogonal plastic flow rule, the proposed model can be used to capture the behaviours of clay with different stiffness. It can also be used to predict existed test results in the literature.

3.1 Analysis on the evolution of yield curve and hardening rule

Parameters for the new Fujinomori clay (NF-clay) [24] as listed in Tab. 1 in section 3.4 are used for demonstration. Two ξ -values: $\xi=2.3$ and $\xi'=1.7$, are used, which correspond to $N=1.19 < M=1.36$ and $N'=1.63 > M$ for the yield curves, as listed in Fig. 5(a) and (b), respectively. Under the same stress step $B_i B_{i+1}$ on triaxial compression stress path with $p=196$ kPa, the evolution process of the yield curves and the hardening rules for two ξ -values are presented in Fig. 5(a) and (b), respectively.

As the stress state changes from B_i to B_{i+1} , as shown in Fig. 5(a), the yield curve changes from the yield curve f_i to the next yield curve f_{i+1} , and the equivalent consolidated pressure point changes from A_i to A_{i+1} accordingly. Since the yield surface is the equivalent surface of the hardening parameter ε_v^p , A_i and B_i (or A_{i+1} and B_{i+1}) on the same yield curve correspond to the same value of ε_v^p , that is, the difference of ε_v^p from B_i to B_{i+1} is just the same with that from A_i to A_{i+1} . The plastic deformation ε_v^p produced by the stress step $B_i B_{i+1}$ can be thus calculated by the change of the equivalent consolidation pressure from A_i to A_{i+1} based on the NCL and the SWL. Based on the evolution of the yield surface and the hardening parameter, the magnitude of the plastic strain increment under the same stress step can thus be calculated. The similar evolution process for $\zeta=1.7$ is show in Fig. 5(b). However, it is obvious that the difference of the equivalent consolidated pressure between A_i and A_{i+1} in Fig. 5(a), i.e., $\Delta p_x = p_{x,i+1} - p_{x,i}$, is bigger than that between A'_i and A'_{i+1} in Fig. 5(b). According to Eq. (9), the change of the hardening parameter for $N=1.19 < M$ is greater than that for $N=1.63 > M$, which leads to a larger magnitude of the plastic strain increment for $N < M$.

(a)

(b)

Fig. 5 Evolution of yield curve and hardening rule for: (a) $\xi=2.3$; (b) $\xi'=1.7$.

3.2 Analysis on the non-orthogonal plastic flow direction

The non-orthogonal plastic flow rule actually plays a key role in the coordination of the yield surface with the critical state concept. Apparently, the plastic flow direction expressed by Eq. (21) is affected by two model parameters μ and N . According to Eqs. (26) and (27), two configurations: (a) $\mu=1.23$ and $N=1.19$ when $\xi=2.3$, and (b) $\mu'=0.57$ and $N'=1.63$ when $\xi'=1.7$, are determined. The triaxial stress pathes A_0B at $p=196$ kPa are the same for these two configurations. The plastic flow direction under the path A_0B can be obtained with the determined parameters, as shown in Fig. 6. The plastic strain increment direction, as indicated by the red arrow for configuration (a) in Fig. 6(a) and the blue arrow for configuration (b) in Fig. 6(b), changes from horizontal to vertical as the stress ratio $\eta=q/p$ increases from 0 to M . For the condition of $\eta=0$ at point A_0 , the plastic flow direction is horizontal to the right, i.e., $d\varepsilon_s^p = 0$, and is independent of μ . With the increase of η , the plastic flow direction changes from horizontal to vertical. When η reaches M at point B , the plastic strain increment is vertically upward, i.e., $d\varepsilon_v^p = 0$, and the critical state is reached, but it is not necessary to be the peak point of the yield curve. For the configuration (a), as shown in Fig. 6(a), the stress state point passes through the peak point of the yield curve and finally reaches the critical state point B that locates on the left of the peak point. For the configuration (b), as shown in Fig. 6(b), the stress state point does not reach the peak point of the yield curve, and the critical state point B locates on the right of the peak point. The comparison of the plastic flow directions for these two configurations is shown in Fig. 7. The orthogonal gradient of the yield curves at point B is also shown by black dashed vectors in Fig. 7 to further illustrate that the critical state point does not necessary to be the peak point of the yield curves.

Fig. 6 Plastic flow direction with fractional-order plastic flow rule for two configurations: (a) $\mu=1.23$ and $N=1.19$ when $\zeta=2.3$; (b) $\mu'=0.57$ and $N'=1.63$ when $\zeta'=1.7$.

Fig. 7 Comparison of the plastic flow directions for two configurations.

The variation rule of the plastic strain increment can also be reflected by the stress-dilatancy relationship d - η , which can be determined based on Eq. (21) and expressed as:

$$d = \frac{d\varepsilon_v^p}{d\varepsilon_s^p} = \frac{\mu N^2 + (\mu^2 - 2\mu)\eta^2}{(3\mu - \mu^2)\eta^{2-\mu}} \quad (28)$$

For the two possible configurations, the stress-dilatancy curves are indicated by red curve and blue curve, respectively. The stress-dilatancy curves for configurations (a) and (b) are below and above the MCC stress-dilatancy curve indicated by the black dashed curve. And eventually, the stress state on the stress-dilatancy curves reaches the same critical state point where η reaches M , as shown in Fig. 8.

Fig. 8 Stress-dilatancy curves.

3.3 Description of soil behaviour with different stiffness

For the conditions that $\mu=0.50, 0.75, 1.00, 1.25$ and 1.50 , the corresponding parameters $N=1.67, 1.52, 1.18, 1.17$ and 0.96 , $\xi=1.67, 1.80, 2.00, 2.33$ and 3.00 are listed in Fig. 9. Under the same stress path A_0B at constant p , the yield curves change from the initial ones to the terminal ones that pass through the critical state point B , as shown in Fig. 9(a). It is obvious that the critical state point B is not the peak point of the yield curves except for the condition of $\mu=1.00$. For different yield curves, the equivalent consolidation pressures are $p_{x1}, p_{x2}, p_{x3}, p_{x4}$ and p_{x5} , respectively. These equivalent consolidation pressures can be used to determine the position of CSLs based on the NCL and the SWL, as shown in Fig. 9(b), which result in a difference in the change of the void ratio for the stress path A_0B .

Fig. 9 Coordination of the critical state with: (a) the yield curve and (b) the hardening rule.

By further introducing the non-orthogonal plastic flow rule presented in section 3.2, the

behaviours of clay with different stiffness are thus captured and shown by the stress-strain curves in Fig. 10. With the increase of μ -value, the predicted principal strains and the plastic volumetric strain at the same stress increment is larger. The prediction result is identical to that of the MCC model when μ is 1. That is, the greater the value of μ , the softer property of clay can be captured. With the change of μ , the behaviours of clay with different stiffness can be reasonably described by the proposed non-orthogonal critical state elastoplastic model.

Fig. 10 Stress-strain relationship with different value of μ .

3.4 Model predictions

To assess the performance of the non-orthogonal critical state model for clay, experimental data from literature are used. The test data of the saturated remoulded normally consolidated NF-clay [24] and the Fujinomori clay (F-clay) [25] under triaxial compression conditions are used for the verification of the non-orthogonal critical state model. Material parameters listed in Tab. 1 from [15] are adopted for model prediction.

Tab. 1 Material parameters used for model prediction

Material parameters	$\lambda/(1+e_0)$	$\kappa/(1+e_0)$	ν	φ	ζ
NF-clay	0.0444	0.0047	0.3	33.7°	2.3
F-clay	0.0508	0.0112	0.3	35°	2.3

Drained triaxial compression tests with $p=196$ kPa on the NF-clay [24] and the F-clay [25] were conducted to explore the strength and deformation properties of saturated remoulded normally consolidated clay. For the NF-clay, test data and material parameters used for prediction are from Chowdhury and Nakai [24]. In this paper, $\zeta=2.3$ can be determined from Eq. (25) based on the void

ratio variation of the NF-clay. On the other hand, the MCC model ($\xi=2.0$) is also used for comparison and demonstration. The fractional order $\mu=1.23$ and 1.00 are determined in accordance with the value of $\xi=2.3$ and $\xi=2$, respectively. And the axial ratio $N=1.19$ and 1.36 ($=M$) can also be determined accordingly based on ξ and M . Model predictions and comparisons with test results of the NF-clay indicated by the red open circles can be found in Fig. 11. Apparently, model predictions with the fractional order $\mu=1.23$ that represented by the red solid line capture the test results better than predictions of the MCC model ($\mu=1.00$) represented by the black dash line. For the F-clay, the test data and material parameters are from Lu *et al.* [15]. Similar variation rule can be found in Fig. 12. As shown in Fig. 11 and Fig. 12, the greater the μ value, the greater the principal strains occurring at the same principal stress ratio. Comparisons and analyses indicate that the non-orthogonal model can be used to well describe the deformation properties of clay with different stiffness.

Fig. 11 Model predictions for test results of the NF-clay (data from [24]) under triaxial compressions.

Fig. 12 Model predictions for test results of the F-clay (data from [25]) under triaxial compressions.

4 Conclusions

In this paper, a non-orthogonal elastoplastic model with critical state is presented, and the magnitude and direction of plastic strain increment are directly determined by the yield function. The proposed non-orthogonal model possesses simple formulations. Only five material parameters with clear physical meanings are needed. They all can be determined by the traditional laboratory tests. The proposed non-orthogonal model can degrade into an orthogonal model, which is the MCC model, when the fractional order μ equals 1. The critical state theory is improved to capture the mechanical behaviours of clay with different stiffness. Moreover, it remains exploratory to be improved to furnish modelling dilatancy of sand.

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